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The q-deformed Moszkowski model: RPA modes

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Abstract. The behaviour of the Moszkowski model within the context of quantum algebras is studied. The Moszkowski Hamiltonian is exactly diagonalized for various values of the deformation parameter of the involved quantum algebra. The phase transition from a vibrational regime to a rotational regime is discussed in terms of the q-deformation. A coherent state for the ground state is introduced, and the Hartree–Fock energy and the RPA frequencies are compared with the exact values. The meaning of the q-deformation in this model is discussed.

1. Introduction

Quantum algebras (also called quantum groups) [1–4], are recently receiving much attention in physics. From the mathematical point of view they are q-deformations of the universal enveloping algebras of the corresponding Lie algebras, being also concrete examples of Hopf algebras [3,4]. When the deformation parameter q is set equal to 1, the usual Lie algebras are obtained. Initially used for solving the quantum Yang–Baxter equation [5], they are now finding applications in several branches of physics, specially after the introduction of the q-deformed harmonic oscillator [6,7]. Applications in conformal field theory, quantum gravity, quantum optics [8,9] as well as in the description of spin chains have already appeared. In nuclear physics it has been found that the pairing correlations in a single-j shell can be described in terms of the usual q-deformed harmonic oscillator approximately [10], or in terms of a generalized q-deformed harmonic oscillator [11] exactly [12], the deformation parameter being related to the inverse of the size of the shell. Furthermore, rotational spectra of deformed [13,14] and superdeformed [15] nuclei and also of diatomic molecules [16,17] have been described in terms of the q-deformed rotator having the symmetry $su_q(2)$. $B(E2)$ transition probabilities in rotational nuclear bands are also well described by the $su_q(2)$ symmetry [18]. In addition, vibrational spectra of diatomic molecules have been described in terms of the q-deformed harmonic [19], anharmonic [20,21] and generalized [22] oscillators. The physical content of the q-deformed harmonic and anharmonic oscillators has been clarified by constructing WKB-equivalent potentials giving the same spectrum as these oscillators [23,24].

From the above-mentioned developments it turns out that rotations and vibrations have been successfully described separately in terms of quantum algebraic symmetries.

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It is therefore reasonable to expect that a common algebraic framework can exist for rotational and vibrational degrees of freedom. It is worth recalling the Moszkowski model [25] at this point, which has an $su(2) \times su(2)$ symmetry. Being a two-dimensional version of the Elliot model, the Moszkowski model is an exactly soluble model in the context of which the rotational and vibrational degrees of freedom can be schematically described, as well as the phase transition from the vibrational to the rotational regime. It is therefore reasonable to consider the q-deformed version of the Moszkowski model and to study the changes induced by the q-deformation on the properties of the model, in particular on the behaviour of the phase transition [26].

In section 2 of the present work, the Moszkowski model is briefly reviewed, while in section 3 its q-deformed version is given. In section 4 the q-analogues of the $su(2)$ Perelomov coherent states [6, 27–31] are introduced and the ground-state and excitation energies are calculated in mean-field approximation, while in section 5 the numerical results are given and the comparison between the exact results and the mean-field treatment is presented. Finally, section 6 contains the main conclusions of the present work and plans for further developments.

2. The Moszkowski model

The Moszkowski model is a two-level model, each of the levels being N-fold degenerate with N_a particles of type a and N_b particles of type b. The state of each particle is specified by the quantum numbers $\sigma = \pm\frac{1}{2}$, taking the value $+\frac{1}{2}$ in the upper level and $-\frac{1}{2}$ in the lower level, and p, which refers to the particular degenerate state within a given level. The $su(2) \times su(2)$ Hamiltonian which describes the model reads

$$H = c(J_z(a) - J_z(b)) + V(J_x^2 + J_y^2) \tag{1}$$

where c is the energy difference between the two levels, V denotes the interaction strength and

$$J_i = J_i(a) + J_i(b) \qquad J^2 = \sum_i J_i^2 \qquad i = x, y, z\,. \tag{2}$$

The quasi-spin operators J_z, J_+ and J_- are defined by

$$J_z(d) = \sum_p \tfrac{1}{2}(d^\dagger_{p,+\frac{1}{2}} d_{p,+\frac{1}{2}} - d^\dagger_{p,-\frac{1}{2}} d_{p,-\frac{1}{2}}) \tag{3a}$$

$$J_+(d) = \sum_p d^\dagger_{p,+\frac{1}{2}} d_{p,-\frac{1}{2}} \tag{3b}$$

$$J_-(d) = \sum_p d^\dagger_{p,-\frac{1}{2}} d_{p,+\frac{1}{2}}, \qquad d = a, b \tag{3c}$$

where the operators $a^\dagger_{p,\pm\frac{1}{2}}$ $\left(b^\dagger_{p,\pm\frac{1}{2}}\right)$ create a particle of type $a\,(b)$ in the state p with $\sigma = \pm\frac{1}{2}$ and $a_{p,\pm\frac{1}{2}}$ $\left(b_{p,\pm\frac{1}{2}}\right)$ are the corresponding annihilation operators.

The operators J_+, J_- and J_z satisfy the usual $su(2)$ commutation relations

$$\begin{aligned}[J_+(i), J_-(j)] &= 2J_z(i)\delta_{ij} \\ [J_z(i), J_\pm(j)] &= \pm J_\pm(i)\delta_{ij} \qquad i,j = a,b\,.\end{aligned} \tag{4}$$

Since

$$[H, J_z(a) + J_z(b)] = [H, J_a^2] = [H, J_b^2] = 0$$

the Moszkowski Hamiltonian is exactly diagonalized in the basis

$$|\psi_{ab}\rangle = \left|\frac{N_a}{2} m_a\right\rangle \left|\frac{N_b}{2} m_b\right\rangle \tag{5}$$

$$m_a = -\frac{N_a}{2}, -\frac{N_a}{2}+1, \ldots, \frac{N_a}{2} \qquad m_b = -\frac{N_b}{2}, -\frac{N_b}{2}+1, \ldots, \frac{N_b}{2}\,. \tag{6}$$

The physical behaviour of the present system is controlled by the Moszkowski parameter $\lambda = N|V|/c$; $\lambda = 0$ corresponds to the vibrational limit and $\lambda = \infty$ to the rotational limit.

3. The deformed Moszkowski model

The $su_q(2)$ quasi-spin operators J_+, J_- and J_z satisfy the commutation relations

$$\begin{aligned}[J_+(i), J_-(j)] &= [2J_z(i)]\delta_{ij} \\ [J_z(i), J_\pm(j)] &= \pm J_\pm(i)\delta_{ij} \qquad i,j = a,b\end{aligned} \tag{7}$$

$$[x] = \frac{q^x - q^{-x}}{q - q^{-1}} \tag{8}$$

where q is the deformation parameter of the algebra. When $q \to 1$, then $[x] = x$. The Casimir operator of the algebra is

$$C_q^2 = J_- J_+ + [J_z][J_z + 1]\,. \tag{9}$$

The application of $J_-(d)$, $J_+(d)$, $J_z(d)$ and $C_q^2(d)$ operators to a generic ket $|j_d m_d\rangle$, which spans the $SU_q(2)$ unitary irreps space [32] gives

$$\begin{aligned}J_z(d)|j_d m_d\rangle &= m_d|j_d m_d\rangle \\ J_+(d)|j_d m_d\rangle &= \sqrt{[j_d - m_d][j_d + m_d + 1]}\,|j_d m_d + 1\rangle \\ J_-(d)|j_d m_d\rangle &= \sqrt{[j_d + m_d][j_d - m_d + 1]}\,|j_d m_d - 1\rangle \\ C_q^2(d)|j_d m_d\rangle &= [j_d][j_d + 1]|j_d m_d\rangle\,.\end{aligned} \tag{10}$$

In terms of these operators we obtain, for the q-deformed Moszkowski Hamiltonian

$$\begin{aligned}H_q = {} & c(J_z(a) - J_z(b)) + V(C_q^2(a) - J_z^2(a) + C_q^2(b) - J_z^2(b) \\ & + J_+(a)J_-(b) + J_+(b)J_-(a))\,.\end{aligned} \tag{11}$$

As in the non-deformed case, the following commutation relations hold:

$$[H_q, J_z] = [H_q, C_q^2(a)] = [H_q, C_q^2(b)] = 0$$

and, therefore, H_q can be diagonalized in the basis

$$|\psi_{ab}\rangle = \left|\frac{N_a}{2} m_a\right\rangle \left|\frac{N_b}{2} m_b\right\rangle .$$

4. Mean-field approximation

We introduce the coherent state defined by

$$|\Psi(\alpha,\beta)\rangle = \exp_q[\alpha J_+(a) + \beta J_-(b)]|0\rangle = \sum_{k,l} \frac{\alpha^k}{[k]!} \frac{\beta^l}{[l]!} J_+^k(a) J_-^l(b)|0\rangle \tag{12}$$

where $|0\rangle = |j_a, -j_a; j_b, j_b\rangle$. The state $|\Psi(\alpha,\beta)\rangle$ is not normalized. The Hartree–Fock ground state is obtained by minimizing

$$E_q(\alpha,\beta) = \frac{\langle\Psi(\alpha,\beta)|H_q|\Psi(\alpha,\beta)\rangle}{\langle\Psi(\alpha,\beta)|\Psi(\alpha,\beta)\rangle} \tag{13}$$

with respect to α, α^* and β, β^*. The equations

$$\frac{\partial}{\partial\alpha} E_q(\alpha,\beta) = \frac{\partial}{\partial\alpha^*} E_q(\alpha,\beta) = 0 \tag{14a}$$

$$\frac{\partial}{\partial\beta} E_q(\alpha,\beta) = \frac{\partial}{\partial\beta^*} E_q(\alpha,\beta) = 0 \tag{14b}$$

have two types of solutions according to the value of the parameter of the model λ:

(a) $\alpha = \beta = 0$ for all values of λ. This corresponds to the non-deformed solution.

(b) $\alpha = \alpha_0$, $\beta = \beta_0$ for λ greater than a critical value λ_c. When this solution exists, it corresponds to a lower energy state than the previous one and, therefore, the system chooses this deformed state.

In figure 3 the broken curves represent the Hartree–Fock energy, for different values of the q-parameter

$$E_{\mathrm{HF}} = E_q(\alpha_0, \beta_0) . \tag{15}$$

The time-dependent Hartree–Fock equations are obtained from the Lagrangian

$$\mathcal{L} = \frac{\mathrm{i}}{2} \frac{\langle\Psi(\alpha,\beta)|\dot\Psi(\alpha,\beta)\rangle - \langle\dot\Psi(\alpha,\beta)|\Psi(\alpha,\beta)\rangle}{\langle\Psi(\alpha,\beta)|\Psi(\alpha,\beta)\rangle} - \frac{\langle\Psi(\alpha,\beta)|H_q|\Psi(\alpha,\beta)\rangle}{\langle\Psi(\alpha,\beta)|\Psi(\alpha,\beta)\rangle} \tag{16}$$

or, introducing the canonically conjugated variables $\tilde\alpha$, $\mathrm{i}\tilde\alpha^*$, $\tilde\beta$, $\mathrm{i}\tilde\beta^*$, defined as

$$\tilde\alpha = \alpha\sqrt{\frac{F_{10}^a}{\alpha\alpha^*}} \qquad \tilde\beta = \beta\sqrt{\frac{F_{10}^b}{\beta\beta^*}} \tag{17}$$

we can write

$$\mathcal{L} = \frac{\mathrm{i}}{2}(\tilde{\alpha}^* \dot{\tilde{\alpha}} - \dot{\tilde{\alpha}}^* \tilde{\alpha} + \tilde{\beta}^* \dot{\tilde{\beta}} - \dot{\tilde{\beta}}^* \tilde{\beta}) - \mathcal{H}(\tilde{\alpha}, \tilde{\alpha}^*, \tilde{\beta}, \tilde{\beta}^*) . \tag{18}$$

In (17) $F^a_{lm}(\alpha)$ is given by

$$F^a_{lm}(\alpha\alpha^*) = \frac{F(l, 2j_a, \alpha\alpha^*)}{F(m, 2j_a, \alpha\alpha^*)}$$

with

$$F(n, 2j_a, \alpha\alpha^*) = \sum_k \frac{[2j_a]!}{[k]![2j_a - k]!} (\alpha\alpha^*)^k k^n \qquad n = 0, 1, 2, \ldots .$$

Similar expressions are considered for particles of the type b. In terms of the new variables the expectation values of the operators $J_z(a)$ and $J_z(b)$ have the form

$$\mathcal{J}_z(a) = -\frac{N_a}{2} + \tilde{\alpha}^* \tilde{\alpha} \qquad \mathcal{J}_z(b) = \frac{N_b}{2} - \tilde{\beta}^* \tilde{\beta} . \tag{19}$$

The expectation values of the operators $J_\pm(a)$ and $J_\pm(b)$ cannot be expressed in a closed form and, in the first order, they are given by

$$\mathcal{J}_+(a) = \sqrt{[2j_a]}\, \tilde{\alpha}^* \qquad \mathcal{J}_-(a) = \sqrt{[2j_a]}\, \tilde{\alpha} \tag{20a}$$

$$\mathcal{J}_+(b) = \sqrt{[2j_b]}\, \tilde{\beta} \qquad \mathcal{J}_-(b) = \sqrt{[2j_b]}\, \tilde{\beta}^* . \tag{20b}$$

In the previous expressions, the following notation has been used:

$$\mathcal{J}_\nu(i) = \frac{\langle \Psi(\alpha, \beta) | J_\nu(i) | \Psi(\alpha, \beta) \rangle}{\langle \Psi(\alpha, \beta) | \Psi(\alpha, \beta) \rangle} \qquad i = a, b \qquad \nu = \pm, z .$$

We will now study the response of the system when small perturbations around the equilibrium state are allowed. In the deformed phase we write

$$\alpha = \alpha_0(1 + \eta) \qquad \beta = \beta_0(1 + \xi) \tag{21a}$$

where α_0, β_0 correspond to the equilibrium values and, for the non-deformed phase,

$$\alpha = \eta \qquad \beta = \xi . \tag{21b}$$

In both cases, η and ξ are small time-dependent quantities.

The time evolution of the coordinates η, ξ can be determined by using the least-action principle, with the Lagrangian given by (16). Expanding $\mathcal{L}$ in powers of η, η^*, ξ, ξ^*, and, introducing the canonical variables $\tilde{\eta}$, $\mathrm{i}\tilde{\eta}^*$, $\tilde{\xi}$, $\mathrm{i}\tilde{\xi}^*$, defined as

$$\tilde{\eta} = \eta\sqrt{\mathcal{N}_a} \qquad \tilde{\xi} = \xi\sqrt{\mathcal{N}_b} \tag{22}$$

where $\mathcal{N}_i = F^i_{20}(\alpha_0) - (F^i_{10}(\alpha_0))^2$, $(i = a, b)$, we obtain, for the second-order Lagrangian,

$$\mathcal{L}^{(2)} = \frac{\mathrm{i}}{2}(\tilde{\eta}^* \dot{\tilde{\eta}} - \dot{\tilde{\eta}}^* \tilde{\eta} + \tilde{\xi}^* \dot{\tilde{\xi}} - \dot{\tilde{\xi}}^* \tilde{\xi}) - \mathcal{H}(\tilde{\eta}, \tilde{\eta}^*, \tilde{\xi}, \tilde{\xi}^*) \tag{23}$$

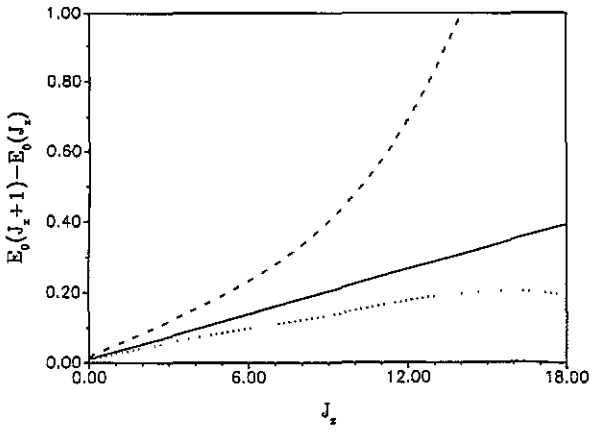

Figure 1. Difference between the energies of two consecutive excited states in the same rotational band versus J_z, in the case $\lambda = 2.0$, for different values of the deformation parameter, $q = \exp(s)$. (Full curve: $s = 0.0$, broken curve: real s ($s = 0.2$) and dotted curve: imaginary s ($s = 0.12i$).

where

$$\mathcal{H}(\tilde{\eta}, \tilde{\eta}^*, \tilde{\xi}, \tilde{\xi}^*) = \tfrac{1}{2}(\tilde{\eta}^* \quad \tilde{\xi}^* \quad \tilde{\eta} \quad \tilde{\xi}) \begin{pmatrix} A & B \\ B & A \end{pmatrix} \begin{pmatrix} \tilde{\eta} \\ \tilde{\xi} \\ \tilde{\eta}^* \\ \tilde{\xi}^* \end{pmatrix}$$

and A and B are 2×2 real and symmetric matrices.

The eigenfrequencies and the eigenvectors describing the time evolution of a mean-field state slightly deviating from the equilibrium state can be obtained by solving the RPA-type equation

$$\begin{pmatrix} A & B \\ B & A \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \Omega \begin{pmatrix} X \\ -Y \end{pmatrix}$$

where X and Y stand for the column matrices

$$X = \begin{pmatrix} \tilde{\eta} \\ \tilde{\xi} \end{pmatrix} \qquad Y = \begin{pmatrix} \tilde{\eta}^* \\ \tilde{\xi}^* \end{pmatrix}$$

and Ω are the RPA frequencies.

The RPA ground-state energy, E_{RPA} [33] is given by

$$E_{\mathrm{RPA}} = E_{\mathrm{HF}} + \tfrac{1}{2}(\Omega_1 + \Omega_2) - \tfrac{1}{2}\mathrm{Tr}A. \tag{24}$$

5. Numerical results

In figues 1 and 2 we represent, respectively for $\lambda = 2.0$ and $\lambda = 0.02$, the difference between the energies of two consecutive excited states in the same rotational band as a function of J_z and for different values of the deformation parameter, $q = \exp(s)$ (full curve: $s = 0.0$, broken curve: real s ($s = 0.2$) and dotted curve: imaginary s ($s = 0.12i$)). For $s = 0.0$ this spacing grows linearly in the rotational regime ($\lambda = 2.0$), or stays almost constant in the vibrational regime ($\lambda = 0.02$). However, for a finite s nonlinear effects are introduced. If s is real, the spacing between two consecutive levels in the same band increases with s (broken curve); however the opposite effect is present for s imaginary: the spacing decreases with s (dotted curve).

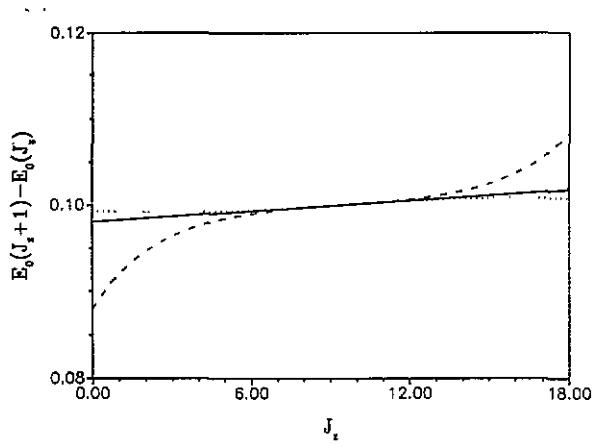

Figure 2. As in figure 1, for $\lambda = 0.02$.

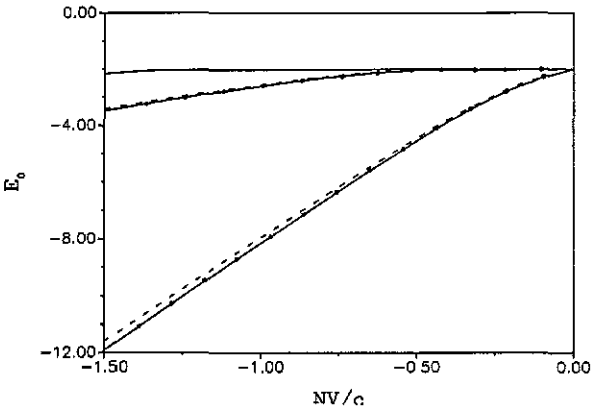

Figure 3. Ground-state energy versus NV/c for different values of the deformation parameter, $q = \exp(s)$. From top to bottom, imaginary s ($s = 0.12$i), $s = 0.0$ and real s ($s = 0.2$), respectively. The full curve corresponds to the exact value, the broken curve to E_{HF} and the circles to E_{RPA}.

In figure 3 we show the ground-state energy as a function of NV/c for different values of the deformation parameter. We have considered $N_a = N_b = 20$ and $c = 0.1$. Curves for $s = 0.0$, real s and imaginary s are displayed. It is seen that the ground-state energy decreases as q increases, for real values of s (lower set of curves); this suggests that an attractive residual interaction is simulated by the q-deformation. On the other hand, when q is a phase (imaginary s), the ground-state energy increases with q (upper curve); we conclude that now a repulsive interaction is introduced by the q-deformation. We also observe a good agreement between the exact results (full curves) and the E_{RPA} (circles) obtained from (24).

We have also investigated the RPA frequencies and the corresponding exact excited-state energies as a function of NV/c. We show the results obtained in figures 4 and 5 for the same choice of parameters concerning figure 3. Two solutions exist corresponding to two types of excitations, namely (i) $E_1(J_z = 1) - E_0(J_z = 0)$ (lower curve) and (ii) $E_1(J_z = 0) - E_0(J_z = 0)$ (upper curve). In the deformed phase the exact solution (i) is very small and the corresponding RPA frequency is zero. This happens because the RPA ground state is not an eigenstate of J_z, but

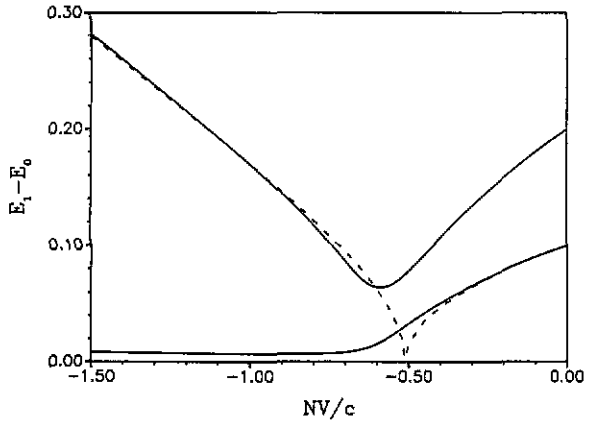

Figure 4. Excitation energy of the first excited state as a function of NV/c for $s = 0.0$ (full curve: exact result; broken curve: RPA approximation).

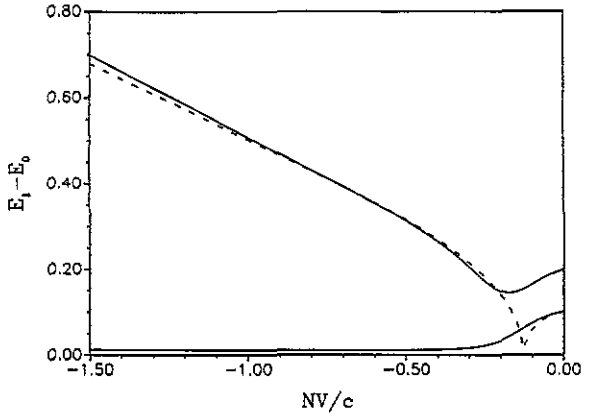

Figure 5. As in figure 4, for $s = 0.2$.

$[H_q, J_z] = 0$. The second exact excited state (ii) in the non-deformed phase is well interpreted as a two-phonon state. In both cases, a clear phase transition from a predominantly rotational behaviour to a vibrational one is identified; we also remark that the phase transition occurs for smaller values of λ as s increases. On the other hand, for an imaginary s, the phase transition is observed for larger values of λ as s increases. This is seen in figure 6, where we represent the exact excitation energy for transitions of the type (i), with imaginary s.

We also note that the RPA results follow very closely the corresponding exact values; this is a clear indication that the mean-field treatment of the problem based on the coherent state introduced in (12) is adequate to study the present q-deformed model.

The usual mean-field and RPA methods describe the dynamics of the system which is obtained as the result of the expansion of the $su_q(2)$ operators in powers of conventional (non-deformed) boson operators.

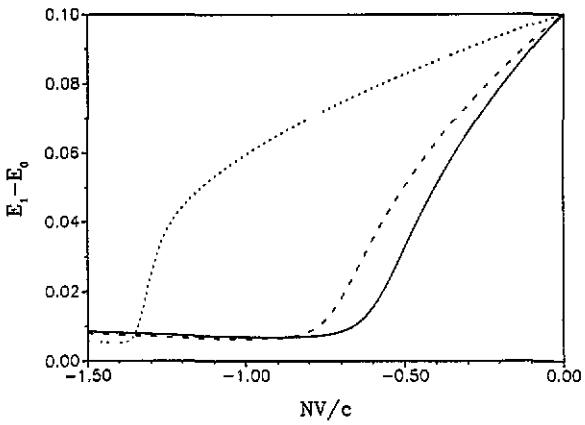

Figure 6. Exact excitation energy of the first excited state (for transitions of the type (i)) as a function of NV/c for imaginary values of s. Full curve: $s = 0.0$, broken curve: $s = 0.06i$ and dotted curve: $s = 0.12i$.

6. Conclusions

In this paper the q-deformed version of the Moszkowski model has been considered and the changes caused on the properties of the model by the q-deformation have been studied, using the usual mean-field and RPA methods. It has been found that the q-deformation simulates an attractive residual interaction, while the phase transition from the vibrational to the rotational behaviour occurs for smaller values of the Moszkowski parameter λ as q increases, in agreement with the findings of [34]. On the other hand, when q is a phase (s imaginary) the effect of the q-deformation corresponds to the introduction of a repulsive residual interaction and the phase transition deviates now for larger values of λ as q increases. The mean-field and RPA approximate results are found to be in good agreement with the exact ones, indicating that the coherent states introduced are adequate for studying the dynamics of the model.

The study of the q-deformation of other exactly solvable models is also of interest. Another model showing a phase transition is the Lipkin model [35] and its generalizations (for a review see [36]). *Quantum*-deformations of algebraic collective models being able to sucessfully describe the experimental data, as the interacting boson model (IBM) [37] (for overviews see [38,39]) should also be studied, since they might allow for improvements in cases in which certain inadequacies appear. A first step in this direction has been taken in [40], where the q-deformation of the vibrational limit of a toy version of the IBM having a $U(3)$ overall symmetry has been studied. Work in these directions is in progress.

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References

[1] Kulish P P and Reshetikhin N Yu 1981 *Zapiski Semenarov LOMI* **101** 101
[2] Sklyanin E K 19892 *Funct. Anal. Appl.* **16** 262

[3] Drinfeld V G 1986 *Proc. Int. Congr. Mathematicians* ed A M Gleason (Providence, RI: American Mathematical Society) p 798
[4] Jimbo M 1986 *Lett. Math. Phys.* **11** 247
[5] Jimbo M 1990 Yang Baxter Equation in Integrable Systems *Advanced Series in Mathematical Physics 10)* (Singapore: World Scientific)
[6] Biedenharn L C 1989 *J. Phys. A: Math. Gen.* **22** L873
[7] Macfarlane A J 1989 *J. Phys. A: Math. Gen.* **22** 4581
[8] Chaichian M, Ellinas D and Kulish P 1990 *Phys. Rev. Lett.* **65** 980
[9] Celegnini E, Rasetti M and Vitiello G 1991 *Phys. Rev. Lett.* **66** 2056
[10] Bonatsos D 1992 *J. Phys. A: Math. Gen.* **25** L101
[11] Daskaloyannis C 1991 *J. Phys. A: Math. Gen.* **24** L789
[12] Bonatsos D and Daskaloyannis C 1992 *Phys. Lett.* **278B** 1
[13] Raychev P P, Roussev R P and Smirnov Yu F 1990 *J. Phys. G: Nucl. Part. Phys.* **16** L137
[14] Bonatsos D, Argyres E N, Drenska S B, Raychev P P, Roussev R P and Smirnov Yu F 1990 *Phys. Lett.* **251B** 477
[15] Bonatsos D, Drenska S B, Raychev P P, Roussev R P and Smirnov Yu F 1991 *J. Phys. G: Nucl. Part. Phys.* **17** L67
[16] Bonatsos D, Raychev P P, Roussev R P and Smirnov Yu F 1990 *Chem. Phys. Lett.* **175** 300
[17] Chang Z and Yan H 1991 *Phys. Lett.* **154A** 254
[18] Bonatsos D, Faessler A, Raychev P P, Roussev R P and Smirnov Yu F 1992 *J. Phys. A: Math. Gen.* **25** 3275
[19] Chang Z and Yan H 1991 *Phys. Rev.* A **43** 6043
[20] Bonatsos D, Raychev P P and Faessler A 1991 *Chem. Phys. Lett.* **178** 221
[21] Bonatsos D, Argyres E N and Raychev P P 1991 *J. Phys. A: Math. Gen.* **24** L403
[22] Bonatsos D and Daskaloyannis C 1992 *Phys. Rev.* A **46** 75
[23] Bonatsos D, Daskaloyannis C and Kokkotas K 1991 *J. Phys. A: Math. Gen.* **24** L795
[24] Bonatsos D, Daskaloyannis C and Kokkotas K 1992 *Chem. Phys. Lett.* **193** 191
[25] Moszkowski S A *Phys. Rev.* 1958 **110** 403
Marshalek E R 1982 *Ann. Phys.* **143** 191
[26] Providência C, Brito L and da Providência J 1992 The q-deformed Moszkowski model *Proc. XIX ICGTMP (Salamanca 1992)*
[27] Perelomov A M 1972 *Commun. Math. Phys.* **26** 222
[28] Kulish P P and Damaskinsky E V 1990 *J. Phys. A: Math. Gen.* **23** L415
[29] Bracken A J, McAnally D S, Zhang R B and Gould M D 1991 *J. Phys. A: Math. Gen.* **24** 1379
[30] Quesne C 1991 *Phys. Lett.* **153A** 303
[31] Buzek V 1991 *J. Mod. Optics* **38** 801
[32] Jimbo M 1985 *Lett. Math. Phys.* **10** 63
[33] Ring P and Schuck P 1980 *The Nuclear Many-Body Problem* (Berlin: Springer)
[34] Menezes D P, Avancini S S and Providência C 1992 *Preprint* IFUSP/P-964 /1992
[35] Lipkin H J, Meshkov N and Glick A J 1965 *Nucl. Phys.* **62** 188
[36] Klein A and Marshalek E R 1991 *Rev. Mod. Phys.* **63** 375
[37] Arima A and Iachello F 1976 *Ann. Phys.* **99** 253; 1978 *Ann. Phys.* **111** 201; 1979 *Ann. Phys.* **123** 468
[38] Iachello F and Arima A 1987 *The Interacting Boson Model* (Cambridge: Cambridge University Press)
[39] Bonatsos D 1988 *Interacting Boson Models of Nuclear Structure* (Oxford: Clarendon)
[40] Bonatsos D, Faessler A, Raychev P P, Roussev R P and Smirnov Yu F 1992 *J. Phys. A: Math. Gen.* **25** L267